

Lösungen

1. LU-Zerlegung:

(Untersuchung und Herleitung im Falle einer 3 x 3-Matrix)

Das Beispiel einer abstrakten 3 x 3-Matrix:

Konstruktion von U (die Elementarsubstitutionen werden in Matrizenoperationen abgebildet):

Sei a_1 nicht 0 (sowie weiter unten auch die Nenner nie 0 - ansonst eine Zeilenvertauschung vorgenommen werden müsste...):

```
Remove["Global`*"]
```

```
A30 = {{a1,b1,c1},{a2,b2,c2},{a3,b3,c3}};
K31 = {{0,0,0},{0,0,0},{A30[[3]][[1]]/A30[[1]][[1]],0,0}};
Map[MatrixForm,{A30, K31}]
```

$$\left\{ \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{a_3}{a_1} & 0 & 0 \end{pmatrix} \right\}$$

```
H13 = IdentityMatrix[3]- K31; H13 // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{a_3}{a_1} & 0 & 1 \end{pmatrix}$$

```
U1 = H13.A30//Simplify; U1 //MatrixForm
```

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 0 & -\frac{a_3 b_1}{a_1} + b_3 & -\frac{a_3 c_1}{a_1} + c_3 \end{pmatrix}$$

```
K21 = {{0,0,0},{U1[[2]][[1]]/U1[[1]][[1]],0,0},{0,0,0}}; K32 // MatrixForm
```

```
K32
```

```
H12 = IdentityMatrix[3]-K21; H12 // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 \\ -\frac{a_2}{a_1} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

U2 = H12.U1//Simplify; U2 //MatrixForm

$$\begin{pmatrix} a1 & b1 & c1 \\ 0 & -\frac{a2b1}{a1} + b2 & -\frac{a2c1}{a1} + c2 \\ 0 & -\frac{a3b1}{a1} + b3 & -\frac{a3c1}{a1} + c3 \end{pmatrix}$$

Sei $-\frac{a2b1}{a1} + b2$ nicht 0:

K32 = {{0,0,0},{0,0,0},{0,U2[[3]][[2]]/U2[[2]][[2]],0}}; K32 // MatrixForm

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{-\frac{a3b1}{a1} + b3}{-\frac{a2b1}{a1} + b2} & 0 \end{pmatrix}$$

H23 = IdentityMatrix[3]-K32; H23 // MatrixForm

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{-\frac{a3b1}{a1} + b3}{-\frac{a2b1}{a1} + b2} & 1 \end{pmatrix}$$

U3 = H23.U2//Simplify; U3 //MatrixForm

$$\begin{pmatrix} a1 & b1 & c1 \\ 0 & -\frac{a2b1}{a1} + b2 & -\frac{a2c1}{a1} + c2 \\ 0 & 0 & \frac{a3b2c1 - a2b3c1 - a3b1c2 + a1b3c2 + a2b1c3 - a1b2c3}{a2b1 - a1b2} \end{pmatrix}$$

In einem Schritt:

U3 = H23.H12.H13.A30//Simplify; U3 //MatrixForm

$$\begin{pmatrix} a1 & b1 & c1 \\ 0 & -\frac{a2b1}{a1} + b2 & -\frac{a2c1}{a1} + c2 \\ 0 & 0 & \frac{a3b2c1 - a2b3c1 - a3b1c2 + a1b3c2 + a2b1c3 - a1b2c3}{a2b1 - a1b2} \end{pmatrix}$$

Uresult = U3;

Konstruktion von L mittels inverser Matrix:

$LU = A \implies L = A \text{ Inverse}U$. Daher ist die Inverse der Dreiecksmatrix U zu bestimmen.

Ansatz: $U * \text{Inverse}U = E$. Die Berechnung von $\text{Inverse}U$ ist hier einfach. Die sukzessive Berechnung der Elemente von $\text{Inverse}U$ zeigt rasch, dass $\text{Inverse}U$ ebenfalls eine obere Dreiecksmatrix sein muss.

U = {{u11,u12,u13},{0,u22,u23},{0,0,u33}}; U // MatrixForm

$$\begin{pmatrix} u11 & u12 & u13 \\ 0 & u22 & u23 \\ 0 & 0 & u33 \end{pmatrix}$$

InvU = {{v11,v12,v13},{0,v22,v23},{0,0,v33}}; InvU // MatrixForm

$$\begin{pmatrix} v11 & v12 & v13 \\ 0 & v22 & v23 \\ 0 & 0 & v33 \end{pmatrix}$$

U.InvU // MatrixForm

$$\begin{pmatrix} u_{11} v_{11} & u_{11} v_{12} + u_{12} v_{22} & u_{11} v_{13} + u_{12} v_{23} + u_{13} v_{33} \\ 0 & u_{22} v_{22} & u_{22} v_{23} + u_{23} v_{33} \\ 0 & 0 & u_{33} v_{33} \end{pmatrix}$$

InvU.U // MatrixForm

$$\begin{pmatrix} u_{11} v_{11} & u_{12} v_{11} + u_{22} v_{12} & u_{13} v_{11} + u_{23} v_{12} + u_{33} v_{13} \\ 0 & u_{22} v_{22} & u_{23} v_{22} + u_{33} v_{23} \\ 0 & 0 & u_{33} v_{33} \end{pmatrix}$$

solv = Solve[U.InvU == IdentityMatrix[3], {v11, v12, v13, v22, v23, v33}] // Flatten

$$\left\{ v_{13} \rightarrow -\frac{u_{13} u_{22} - u_{12} u_{23}}{u_{11} u_{22} u_{33}}, v_{11} \rightarrow \frac{1}{u_{11}}, v_{12} \rightarrow -\frac{u_{12}}{u_{11} u_{22}}, v_{22} \rightarrow \frac{1}{u_{22}}, v_{23} \rightarrow -\frac{u_{23}}{u_{22} u_{33}}, v_{33} \rightarrow \frac{1}{u_{33}} \right\}$$

Hier hat man ein lineares Gleichungssystem mit 6 Unbekannten, das man nach dem Rückwärtseinsetzungsverfahren rasch und problemlos lösen kann.

InvU = InvU /. solv; InvU // MatrixForm

$$\begin{pmatrix} \frac{1}{u_{11}} & -\frac{u_{12}}{u_{11} u_{22}} & -\frac{u_{13} u_{22} - u_{12} u_{23}}{u_{11} u_{22} u_{33}} \\ 0 & \frac{1}{u_{22}} & -\frac{u_{23}}{u_{22} u_{33}} \\ 0 & 0 & \frac{1}{u_{33}} \end{pmatrix}$$

Kontrolle:

Inverse[U] // MatrixForm

$$\begin{pmatrix} \frac{1}{u_{11}} & -\frac{u_{12}}{u_{11} u_{22}} & -\frac{u_{13} u_{22} + u_{12} u_{23}}{u_{11} u_{22} u_{33}} \\ 0 & \frac{1}{u_{22}} & -\frac{u_{23}}{u_{22} u_{33}} \\ 0 & 0 & \frac{1}{u_{33}} \end{pmatrix}$$

Ersetzung der künstlichen Koeffizienten in U durch die von A30:

UFlat = Flatten[U];

U3Flat = Flatten[U3];

rul = Table[UFlat[[k]] -> U3Flat[[k]], {k, 1, Length[UFlat]}]

$$\left\{ u_{11} \rightarrow a_1, u_{12} \rightarrow b_1, u_{13} \rightarrow c_1, 0 \rightarrow 0, u_{22} \rightarrow -\frac{a_2 b_1}{a_1} + b_2, u_{23} \rightarrow -\frac{a_2 c_1}{a_1} + c_2, \right. \\ \left. 0 \rightarrow 0, 0 \rightarrow 0, u_{33} \rightarrow \frac{a_3 b_2 c_1 - a_2 b_3 c_1 - a_3 b_1 c_2 + a_1 b_3 c_2 + a_2 b_1 c_3 - a_1 b_2 c_3}{a_2 b_1 - a_1 b_2} \right\}$$

L = A30.InvU; Lresult = L /. rul // Simplify; Lresult // MatrixForm

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{a_2}{a_1} & 1 & 0 \\ \frac{a_3}{a_1} & \frac{a_3 b_1 - a_1 b_3}{a_2 b_1 - a_1 b_2} & 1 \end{pmatrix}$$

Kontrolle:

```
Lresult.Uresult //Simplify // MatrixForm
```

$$\begin{pmatrix} a1 & b1 & c1 \\ a2 & b2 & c2 \\ a3 & b3 & c3 \end{pmatrix}$$

```
Simplify[Lresult.Uresult] == A30
```

```
True
```

0. Materialbereitstellung

```
A2 = {{1,2},{1,-1}};
```

```
B2 = {{3,5},{4,7}};
```

```
C2 = {{4,1},{5,6}};
```

```
Map[MatrixForm,{A2,B2,C2}]
```

$$\left\{ \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 3 & 5 \\ 4 & 7 \end{pmatrix}, \begin{pmatrix} 4 & 1 \\ 5 & 6 \end{pmatrix} \right\}$$

```
A3 = {{1,2,3},{1,-1,0},{-2,1,5}};
```

```
B3 = {{3,5,1},{4,7,9},{3,2,6}};
```

```
C3 = {{4,1,3},{5,6,5},{5,8,8}};
```

```
Map[MatrixForm,{A3,B3,C3}]
```

$$\left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & -1 & 0 \\ -2 & 1 & 5 \end{pmatrix}, \begin{pmatrix} 3 & 5 & 1 \\ 4 & 7 & 9 \\ 3 & 2 & 6 \end{pmatrix}, \begin{pmatrix} 4 & 1 & 3 \\ 5 & 6 & 5 \\ 5 & 8 & 8 \end{pmatrix} \right\}$$

```
A4 = {{1,2,3,4},{1,-1,0,1},{-2,1,5,2},{-2,2,1,5}};
```

```
B4 = {{3,5,1,1},{4,7,9,5},{3,2,6,8},{5,6,5,1}};
```

```
Map[MatrixForm,{A4,B4}]
```

$$\left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & -1 & 0 & 1 \\ -2 & 1 & 5 & 2 \\ -2 & 2 & 1 & 5 \end{pmatrix}, \begin{pmatrix} 3 & 5 & 1 & 1 \\ 4 & 7 & 9 & 5 \\ 3 & 2 & 6 & 8 \\ 5 & 6 & 5 & 1 \end{pmatrix} \right\}$$

```
A24 = {{3,5,1,1},{4,7,9,5}};
```

```
B42 = {{1,2},{1,-1},{-2,1},{-2,2}};
```

```
Map[MatrixForm,{A24,B42}]
```

$$\left\{ \begin{pmatrix} 3 & 5 & 1 & 1 \\ 4 & 7 & 9 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & -1 \\ -2 & 1 \\ -2 & 2 \end{pmatrix} \right\}$$

```
X13 = {{x11,x12,x13}};
```

```
X31 = {{x11},{x21},{x31}};
```

```
X24 = {{x11,x12,x13,x14},{x21,x22,x23,x24}};
```

```
X42 = {{x11,x12},{x21,x22},{x31,x32},{x41,x42}};
```

```
Map[MatrixForm,{X13,X31,X24,X42}]
```

$$\left\{ (x11 \ x12 \ x13), \begin{pmatrix} x11 \\ x21 \\ x31 \end{pmatrix}, \begin{pmatrix} x11 & x12 & x13 & x14 \\ x21 & x22 & x23 & x24 \end{pmatrix}, \begin{pmatrix} x11 & x12 \\ x21 & x22 \\ x31 & x32 \\ x41 & x42 \end{pmatrix} \right\}$$

```

b31 = {{50},{-100},{1000}};
b32 = {{50,203},{-100,105},{1000,-50}}; ;
Map[MatrixForm,{b31,b32}]

```

$$\left\{ \begin{pmatrix} 50 \\ -100 \\ 1000 \end{pmatrix}, \begin{pmatrix} 50 & 203 \\ -100 & 105 \\ 1000 & -50 \end{pmatrix} \right\}$$

```

Em2 = IdentityMatrix[2];
Em3 = IdentityMatrix[3];
Em4 = IdentityMatrix[4];
Map[MatrixForm,{Em2,Em3,Em4}]

```

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}$$

```

NullMatrix[m_]:= Table[ Table[0,{k,1,m}],{k,1,m}];
NullMatrix[4]//MatrixForm

```

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

```

ABC = {{a,b,c,d,e},{f,g,h,i,j},{k,l,m,n,o},{p,q,r,s,t},{u,v,w,x,y}}; ABC //
MatrixForm

```

$$\begin{pmatrix} a & b & c & d & e \\ f & g & h & i & j \\ k & l & m & n & o \\ p & q & r & s & t \\ u & v & w & x & y \end{pmatrix}$$

```

VdM4 = {{1,2,3,4},{2,3,4,5},{-3,-2,-1,0},{3,4,5,6}}; VdM4 // MatrixForm

```

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ -3 & -2 & -1 & 0 \\ 3 & 4 & 5 & 6 \end{pmatrix}$$

1. LU-Zerlegung für 3 x 3 Matrizen als Modul

(Einfach lesbarer Modul ohne kompakte Programmierung, wiederholt anwendbar!)

```

modulLU[{{a1_,b1_,c1_},{a2_,b2_,c2_},{a3_,b3_,c3_}}]:=
Modul[{}],
A30 = {{a1,b1,c1},{a2,b2,c2},{a3,b3,c3}};
K31 = {{0,0,0},{0,0,0},{A30[[3]][[1]]/A30[[1]][[1]],0,0}};
Map[MatrixForm,{A30, K31}];
H13 = IdentityMatrix[3]- K31;
U1 = H13.A30//Simplify;
K21 = {{0,0,0},{U1[[2]][[1]]/U1[[1]][[1]],0,0},{0,0,0}};
H12 = IdentityMatrix[3]-K21;
U2 = H12.U1//Simplify;
K32 = {{0,0,0},{0,0,0},{0,U2[[3]][[2]]/U2[[2]][[2]],0}};
H23 = IdentityMatrix[3]-K32;
U3 = H23.U2//Simplify;
Uresult = U3;
U = {{u11,u12,u13},{0,u22,u23},{0,0,u33}};
InvU = {{v11,v12,v13},{0,v22,v23},{0,0,v33}};
solv = Solve[U.InvU == IdentityMatrix[3],{v11,v12,v13,v22,v23,v33}] // Flatten;
InvU = InvU /.solv;
UFlat = Flatten[U];
U3Flat = Flatten[U3];
rul = Table[UFlat[[k]]->U3Flat[[k]],{k,1,Length[UFlat]};]
L = A30.InvU; Lresult = L/.rul //Simplify;
Print["Eingabematrix = ",A30//MatrixForm];
Print["U = ",Uresult//MatrixForm];
Print["L = ",Lresult//MatrixForm];
Print["Kontrolle: L U = ",Lresult.Uresult//MatrixForm];]

```

modulLU[A3];

$$\text{Eingabematrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & -1 & 0 \\ -2 & 1 & 5 \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -3 \\ 0 & 0 & 6 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & -\frac{5}{3} & 1 \end{pmatrix}$$

$$\text{Kontrolle: } L U = \begin{pmatrix} 1 & 2 & 3 \\ 1 & -1 & 0 \\ -2 & 1 & 5 \end{pmatrix}$$

modulLU[B3];

$$\text{Eingabematrix} = \begin{pmatrix} 3 & 5 & 1 \\ 4 & 7 & 9 \\ 3 & 2 & 6 \end{pmatrix}$$

$$U = \begin{pmatrix} 3 & 5 & 1 \\ 0 & \frac{1}{3} & \frac{23}{3} \\ 0 & 0 & 74 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{4}{3} & 1 & 0 \\ 1 & -9 & 1 \end{pmatrix}$$

$$\text{Kontrolle: } L U = \begin{pmatrix} 3 & 5 & 1 \\ 4 & 7 & 9 \\ 3 & 2 & 6 \end{pmatrix}$$

modulLU[C3];

$$\text{Eingabematrix} = \begin{pmatrix} 4 & 1 & 3 \\ 5 & 6 & 5 \\ 5 & 8 & 8 \end{pmatrix}$$

$$U = \begin{pmatrix} 4 & 1 & 3 \\ 0 & \frac{19}{4} & \frac{5}{4} \\ 0 & 0 & \frac{47}{19} \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{5}{4} & 1 & 0 \\ \frac{5}{4} & \frac{27}{19} & 1 \end{pmatrix}$$

$$\text{Kontrolle: } L U = \begin{pmatrix} 4 & 1 & 3 \\ 5 & 6 & 5 \\ 5 & 8 & 8 \end{pmatrix}$$

2. Determinantenberechnungen

(Berechnung zur Uebung von Hand bis und mit 4 x 4-Matrizen)

Det[A2]

-3

{Det[A2], Det[B2], Det[C2]}

{-3, 1, 19}

{Det[A3], Det[B3], Det[C3]}

{-18, 74, 47}

{Det[A4], Det[B4]}

{-111, -430}

Det[A24]

Det::matsq : Argument {{3, 5, 1, 1}, {4, 7, 9, 5}} at position 1 is not a nonempty square matrix. Mehr...

Det[{{3, 5, 1, 1}, {4, 7, 9, 5}}]

Geht nicht!

Det[B42]

Det::matsq :

Argument {{1, 2}, {1, -1}, {-2, 1}, {-2, 2}} at position 1 is not a nonempty square matrix. Mehr...

Det[{{1, 2}, {1, -1}, {-2, 1}, {-2, 2}}]

Geht nicht!

Det[X13]

Det::matsq : Argument {{x11, x12, x13}} at position 1 is not a nonempty square matrix. Mehr...

Det[{{x11, x12, x13}}]

Det[X31]

Det::matsq : Argument {{x11}, {x21}, {x31}} at position 1 is not a nonempty square matrix. Mehr...

Det[{{x11}, {x21}, {x31}}]

Det[X24]

Det::matsq :

Argument {{x11, x12, x13, x14}, {x21, x22, x23, x24}} at position 1 is not a nonempty square matrix. Mehr...

Det[{{x11, x12, x13, x14}, {x21, x22, x23, x24}}]

Det[X42]

Det::matsq : Argument {{x11, x12}, {x21, x22}, {x31, x32}, {x41, x42}}
at position 1 is not a nonempty square matrix. Mehr...

Det[{{x11, x12}, {x21, x22}, {x31, x32}, {x41, x42}}]

Det[b31]

Det::matsq : Argument {{50}, {-100}, {1000}} at position 1 is not a nonempty square matrix. Mehr...

Det[{{50}, {-100}, {1000}}]

Det[b32]

Det::matsq :

Argument {{50, 203}, {-100, 105}, {1000, -50}} at position 1 is not a nonempty square matrix. Mehr...

Det[{{50, 203}, {-100, 105}, {1000, -50}}]

{Det[Em2], Det[Em3], Det[Em4], Det[NullMatrix[4]]}

{1, 1, 1, 0}

Det[ABC]

eimqu-djmqu-ehnu+cjnqu+dhoqu-cioqu-eilru+djlru+egnru-
bjnru-dgoru+bioru+ehlsu-cjlsu-egmsu+bjmsu+cgosu-bhosu-
dhltu+ciltu+dgmtu-bimtu-cgntu+bhntu-eimpv+djmpv+
ehnpv-cjnpv-dhopv+ciopv+eikrv-djkrv-efnrv+ajnr+dforv-
aiorv-ehksv+cjksv+efmsv-ajmsv-cfosv+ahosv+dhktv-ciktv-
dfmtv+aimtv+cfntv-ahntv+eilpw-djlpw-egnpw+bjnpw+dgopw-
biopw-eikqw+djkqw+efnqw-ajnqw-dfoqw+aioqw+egksw-bjksw-
eflsw+ajlsw+bfosw-agosw-dgktw+biktw+dfltw-ailtw-bfntw+
agntw-ehlpw+cjlpw+egmpw-bjmpw-cgopw+bhopw+ehkqx-
cjkqx-efmqx+ajmqx+cfqox-ahqox-egkrx+bjkrx+eflrx-
ajlrx-bforx+agorx+cgktx-bhktx-cfltx+ahltx+bfmtx-
agmtx+dhlpw-cilpy-dgmpy+bimpy+cgmpy-bhmpy-dhkqy+
cikqy+dfmqy-aimqy-cfnqy+ahnqy+dgkry-bikry-dflry+
ailry+bfnry-agnry-cgksy+bhksy+cflsy-ahlsy-bfmsy+agmsy

Det[VdM4]

0