Lösungen

Teil 1: Repetition von Begriffen

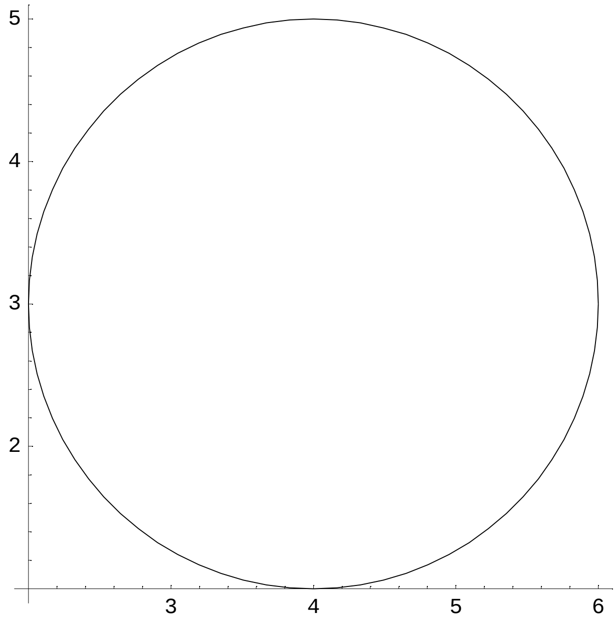
1 Kreis, Kugel

Beispiele von Plots (Formeln im Skript)

```
Remove["Global`*"]
```

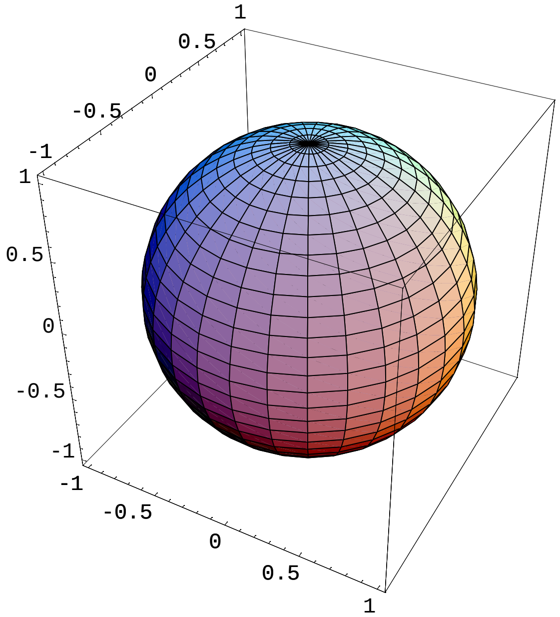
Kreisgleichung und Kugelgleichung: Siehe Skript.

```
{xM,yM}={4,3};   r=2;

k[x_,y_]:= ({x,y}-{xM,yM}).({x,y}-{xM,yM})/r^2;
<< Graphics`ImplicitPlot`;
ImplicitPlot[k[x,y] == 1, {x, 2,6}];
```



Hier die Kugel in Parameterdarstellung (Kugelkoordinaten):

```
ParametricPlot3D[{Cos[φ] Sin[ξ],Sin[φ] Sin[ξ],Cos[ξ]},{φ,0,2 π},{ξ,0,π}];
```



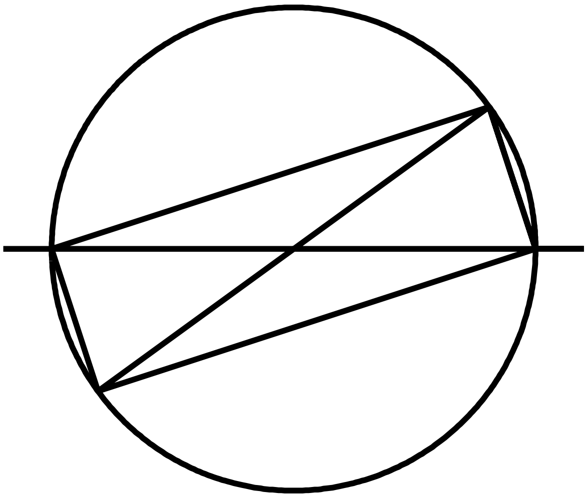
2 Thaleskreis

Beispiel des Plots (Formeln im Skript)

```
Remove["Global`*"]
```

Betrachte die Skizze. Die Lösung sollte sichtbar werden. Der Thaleskreis bei einem rechtwinkligen Dreieck kann als Umkreis des Rechtecks gedeutet werden, das man erhält, wenn man das Dreieck am Mittelpunkt seiner Hypothenuse spiegelt. Mit dem Umkreis ist alles sehr einfach erklärbar!

```
Show[Graphics[{Thickness[0.01], Circle[{0, 0}, 1],
    Line[{{-1.2, 0}, {1.2, 0}, {1, 0},
      {Cos[Pi / 5], Sin[Pi / 5]}, {-1, 0}, -{Cos[Pi / 5], Sin[Pi / 5]},
      {1, 0}, {Cos[Pi / 5], Sin[Pi / 5]}, -{Cos[Pi / 5], Sin[Pi / 5]}}]
   }], AspectRatio → Automatic];
```



3 Apolloniuskreis

Beispiele von Plots (Formeln im Skript)

```
Remove["Global`*"]

p1={0,0}; p2={15,0};
p3={x1,0};
solv1=Solve[x1/(15-x1)==2,{x1}]//Flatten
```

```
{x1 → 10}
```

```
p3={x1,0}/.solv1; p3
```

```
{10, 0}
```

```
p4={x2,0};
solv2=Solve[x2/(x2-15)==2,{x2}]//Flatten
```

```
{x2 → 30}
```

```
p4={x2,0}/.solv2; p4
```

```
{30, 0}
```

```
{xM,yM}=(p3+p4)/2
```

```
{20, 0}
```

```
r=((p4-p3)/2)[[1]]
```

```
10
```

```
k[x_,y_]:= ({x,y}-{xM,yM}).({x,y}-{xM,yM})/r^2;
<< Graphics`ImplicitPlot`;
pl1=ImplicitPlot[k[x,y] == 1, {x, 0,30},DisplayFunction->Identity];
```

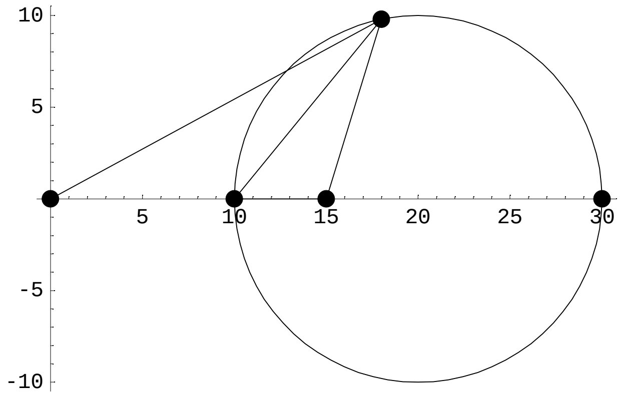
```
solv3=Solve[k[18,y]==1,{y}]//Flatten
```

$\{y \to -4\sqrt{6},\ y \to 4\sqrt{6}\}$

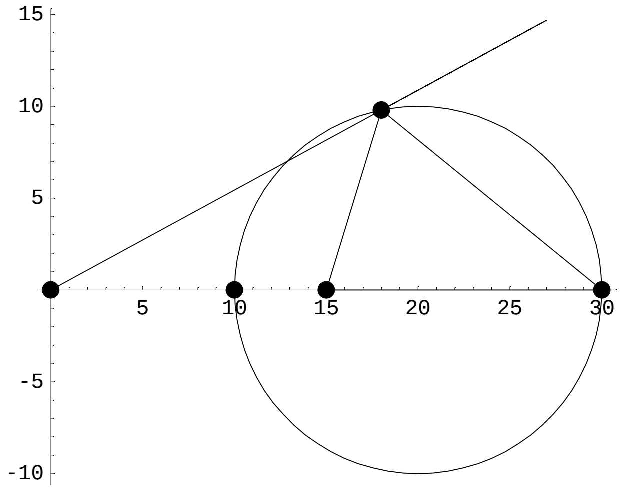
```
p5={18,y}/.solv3[[2]]
```

$\{18,\ 4\sqrt{6}\}$

```
Show[pl1,Graphics[{PointSize[0.03],Point[p1],Point[p2],Point[p3],Point[p4],Point[p5
],Line[{p1,p5,p3,p2,p5}]}],DisplayFunction->$DisplayFunction];
```




```
Show[pl1,Graphics[{PointSize[0.03],Point[p1],Point[p2],Point[p3],Point[p4],Point[p5
],Line[{p1,1.5 p5,p5,p4,p2,p5}]}],DisplayFunction->$DisplayFunction];
```



4 Kegelschnitte

a) Ellipse: Beispiele von Plots (Formeln im Skript)

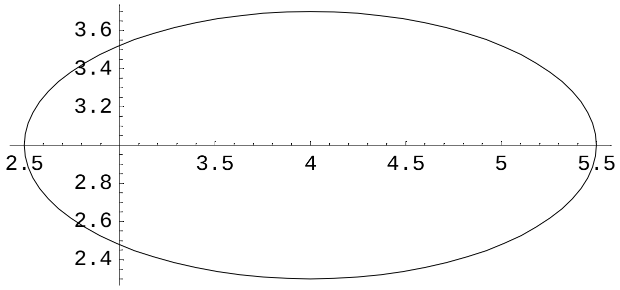
```
Remove["Global`*"]
```

Ellipse

```
{xM,yM}={4,3};   a=1.5;  b=0.7;

k[x_,y_]:= (x-xM)^2/a^2+(y-yM)^2/b^2;  k[x,y]
```

$0.444444\,(-4+x)^2+2.04082\,(-3+y)^2$

```
<< Graphics`ImplicitPlot`;
ImplicitPlot[k[x,y] == 1, {x, 2,6}];
```



Hyperbel: Beispiele von Plots (Formeln im Skript)

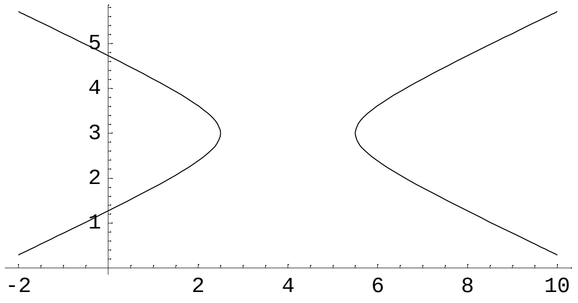
```
Remove["Global`*"]
```

Ellipse

```
{xM,yM}={4,3};   a=1.5;  b=0.7;

k[x_,y_]:= (x-xM)^2/a^2-(y-yM)^2/b^2;  k[x,y]
```

$0.444444\,(-4+x)^2-2.04082\,(-3+y)^2$

```
<< Graphics`ImplicitPlot`;
ImplicitPlot[k[x,y] == 1, {x, -2,10}];
```



Parabel: Beispiele von Plots (Formeln im Skript)

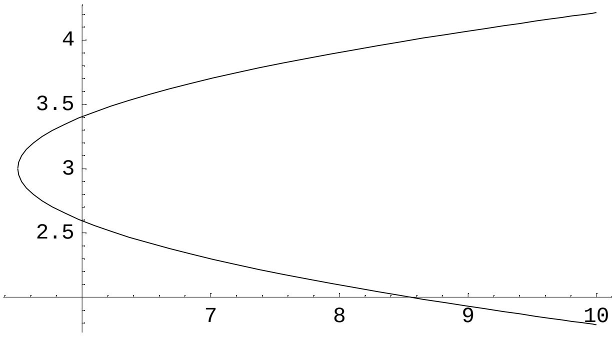
```
Remove["Global`*"]
```

Ellipse

```
{xM,yM}={4,3};   a=1.5;  b=0.7;

k[x_,y_]:= (x-xM)/a-(y-yM)^2/b^2;  k[x,y]
```

$0.666667\,(-4+x)-2.04082\,(-3+y)^2$

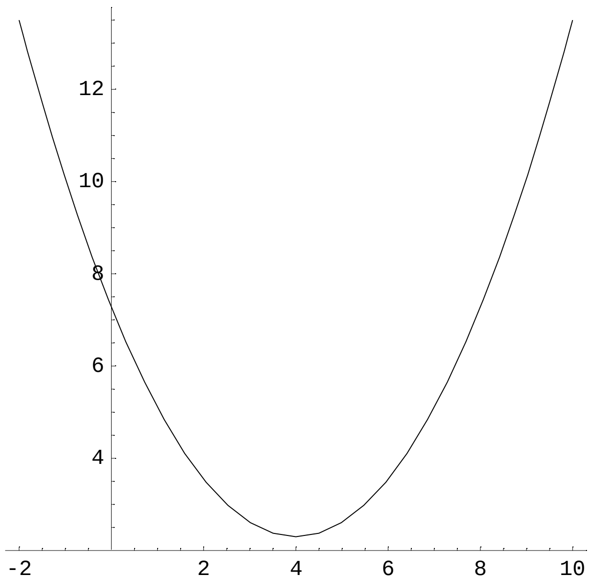
```
<< Graphics`ImplicitPlot`;
ImplicitPlot[k[x,y] == 1, {x, -2,10}];
```




```
{xM,yM}={4,3};   a=1.5;  b=0.7;

k[x_,y_]:= (x-xM)^2/a^2-(y-yM)/b;  k[x,y]
```

$0.444444\,(-4+x)^2-1.42857\,(-3+y)$

```
<< Graphics`ImplicitPlot`;
ImplicitPlot[k[x,y] == 1, {x, -2,10}];
```



5 Tangente und Tangentialebene

a) Tangente: Beispiele von Plots (Formeln im Skript)

```
Remove["Global`*"]
```

```
{xM,yM}={4,3};   r=2;
k[x_,y_]:= ({x,y}-{xM,yM}).({x,y}-{xM,yM})/r^2;
tT = {5.2,y};
solv4=Solve[k[5.2,y]==1,{y}]//Flatten
```

$\{y \to 1.4,\ y \to 4.6\}$

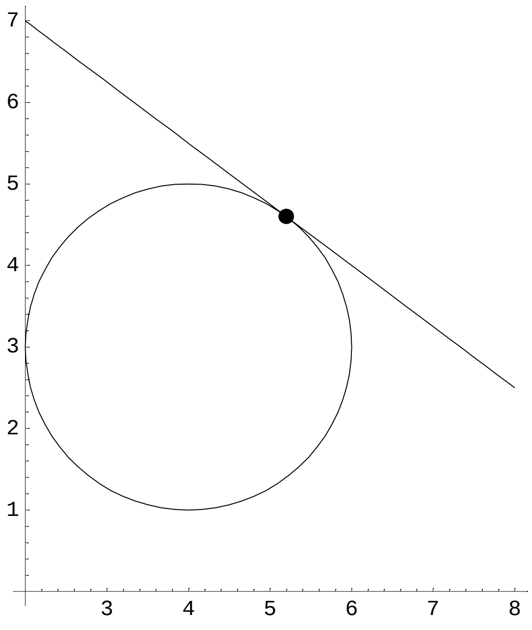
```
tT = {5.2,y}/.solv4[[2]]
```

$\{5.2,\ 4.6\}$

```
tang[x_,y_]:= ({x,y}-{xM,yM}).(tT-{xM,yM})/r^2;
tang[x,y]
```

$\frac{1}{4}\,(1.2\,(-4+x)+1.6\,(-3+y))$

```
<< Graphics`ImplicitPlot`;
pl1 = ImplicitPlot[k[x,y] == 1, {x, 2,8},DisplayFunction->Identity];
pl2 = ImplicitPlot[tang[x,y] == 1, {x, 2,8},DisplayFunction->Identity];

Show[pl1,pl2,Graphics[{PointSize[0.03],Point[tT]}],DisplayFunction->$DisplayFunctio
n];
```



b) Tangentialebene: Beispiele von Plots (Formeln im Skript)

```
Remove["Global`*"]

{xM,yM,zM}={4,3,2.5};   r=2;
k[x_,y_,z_]:= ({x,y,z}-{xM,yM,zM}).({x,y,z}-{xM,yM,zM})/r^2;
tT = {5,3.8,z};
solv4=Solve[k[5,3.8,z]==1,{z}]//Flatten
```

$\{z \to 0.963771,\ z \to 4.03623\}$

```
solv5=Solve[k[x,y,z]==1,z]//Flatten
```

$$\{z \to 2.\ (1.25 - 0.5\sqrt{-21. + 8.\ x - 1.\ x^2 + 6.\ y - 1.\ y^2}),\ z \to 2.\ (1.25 + 0.5\sqrt{-21. + 8.\ x - 1.\ x^2 + 6.\ y - 1.\ y^2})\}$$

```
kugel[x_,y_]:=z/.solv5[[2]];  kugel[z]
```

kugel[z]

```
tT = {5,3.8,z}/.solv4[[2]]
```

{5, 3.8, 4.03623}

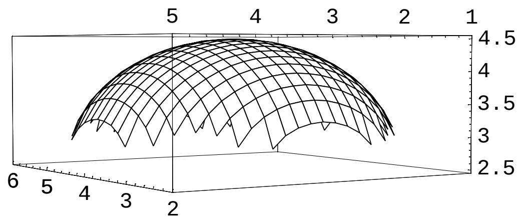
```
tang[x_,y_,z_]:= ({x,y,z}-{xM,yM,zM}).(tT-{xM,yM,zM})/r^2;
tang[x,y,z]
```

$$\frac{1}{4}\ (-4 + x + 0.8\ (-3 + y) + 1.53623\ (-2.5 + z))$$

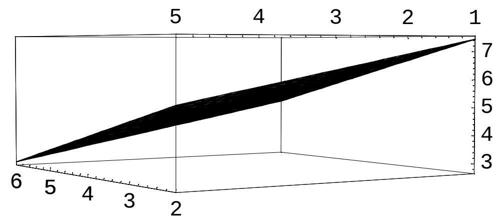
```
solv6=Solve[tang[x,y,z]==1,z]//Flatten
```

$$\{z \to 2.60378\ (2.96014 - 0.25\ x - 0.2\ (-3. + y))\}$$

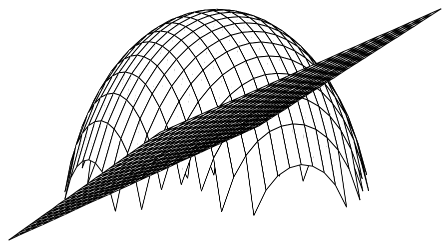
```
ebene[x_,y_]:=z/.solv6

pl1 = Plot3D[Evaluate[kugel[x,y]],{x, 2,6},{y,1,5},ViewPoint->{-3.611, 2.291, 0.170},Shading→False];
```



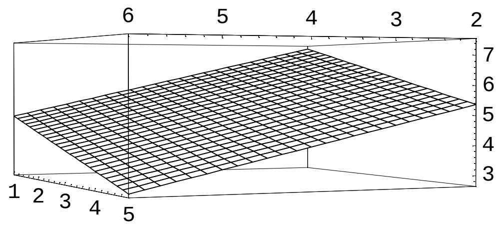

```
pl2 = Plot3D[Evaluate[ebene[x,y]],{x, 2,6},{y,1,5},ViewPoint->{-3.611, 2.291, 0.170},Shading→False];
```



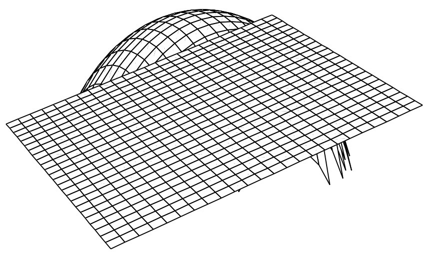

```
Show[pl1,pl2,Graphics[{PointSize[0.03],Point[tT]}]];
```



```
pl2 = Plot3D[Evaluate[ebene[x,y]],{x, 2,6},{y,1,5},ViewPoint->{1.792, 3.990, 0.084},Shading→False];
```




```
Show[pl1,pl2,Graphics[{PointSize[0.03],Point[tT]}]];
```



6 und 7 Pol und Polare

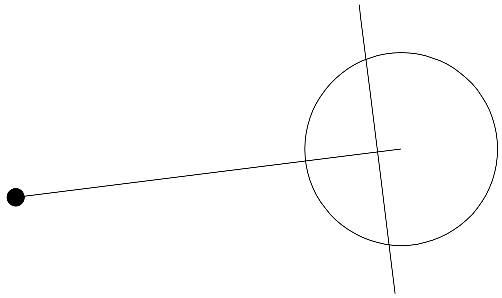
```
Remove["Global`*"]

{xM,yM}={4,3};   r=2;   pM={xM,yM};
k[x_,y_]:= ({x,y}-{xM,yM}).({x,y}-{xM,yM})/r^2;
pol={-4,2}
```

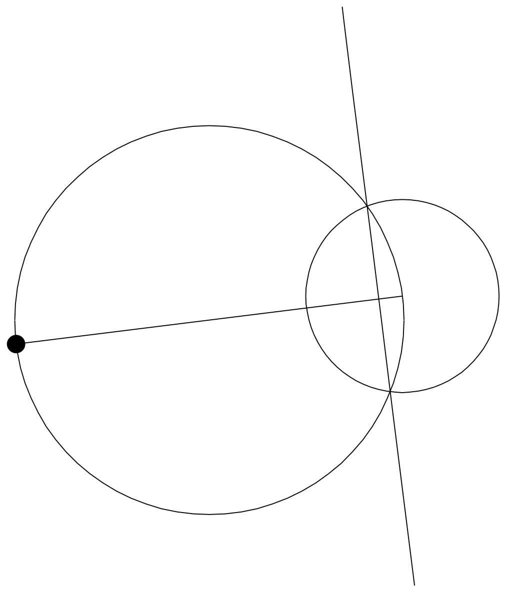
{-4, 2}

```
polare[x_,y_]:= ({x,y}-{xM,yM}).(pol-{xM,yM})/r^2;
polare[x,y]
```

$$\frac{1}{4}(3-8(-4+x)-y)$$

```
pPolM=(pM+pol)/2; rM=Norm[pPolM-pol];
kM[x_,y_]:= ({x,y}-pPolM).({x,y}-pPolM)/rM^2;

<< Graphics`ImplicitPlot`;
pl1 = ImplicitPlot[k[x,y] == 1, {x, 2,8},DisplayFunction->Identity];
pl2 = ImplicitPlot[polare[x,y] == 1, {x,
2,8},DisplayFunction->Identity,PlotRange->{0,6}];
pl3 = ImplicitPlot[kM[x,y] == 1, {x,
-5,8},DisplayFunction->Identity,PlotRange->{-7,12}];


Show[Graphics[{PointSize[0.03],Point[pol],Line[{pol,pM}]}],pl1,pl2,DisplayFunction->$DisplayFunction,PlotRange->{0,6},AspectRatio->Automatic];

Show[Graphics[{PointSize[0.03],Point[pol],Line[{pol,pM}]}],pl1,pl2,pl3,DisplayFunction->$DisplayFunction,PlotRange->{-3,9},AspectRatio->Automatic];
```





Die Polare kann man verwenden zur Tangentenkonstruktion vom Pol aus!

8 und 9   Potenz und Potenzgerade

```
Remove["Global`*"]
```

a Potenz berechnen zu einem Kreis und einem Punkt P0

```
{xM,yM}={4,3};   r=2;
k[x_,y_]:= ({x,y}-{xM,yM}).({x,y}-{xM,yM})/r^2;
k[{x_,y_}]:= k[x,y];
P0={-1,2};
potenz[Pk_]:=k[Pk];
potenz[P0]
```

$\frac{13}{2}$

```
N[%]
```

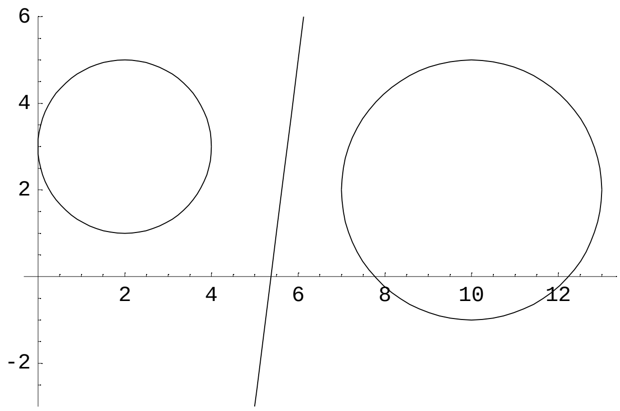
6.5

a Potenzgerade: Gerade gleicher Potenz zu zwei Kreisen

```
{xM1,yM1}={2,3};   r1=2;
k1[x_,y_]:= ({x,y}-{xM1,yM1}).({x,y}-{xM1,yM1})-r1^2;
k1[{x_,y_}]:= k1[x,y];

{xM2,yM2}={10,2};   r2=3;
k2[x_,y_]:= ({x,y}-{xM2,yM2}).({x,y}-{xM2,yM2})-r2^2;
k2[{x_,y_}]:= k2[x,y];

potenzgerade[x_,y_]:=k1[x,y]-k2[x,y]//Simplify;
potenzgerade[x,y]
```

$16\,x-2\,(43+y)$

```
<< Graphics`ImplicitPlot`;
pl1 = ImplicitPlot[k1[x,y] == 0, {x, -1,6},DisplayFunction->Identity];
pl2 = ImplicitPlot[k2[x,y] == 0, {x,
5,14},DisplayFunction->Identity,PlotRange->{-3,6}];
pl3 = ImplicitPlot[potenzgerade[x,y] == 0, {x,
4,8},DisplayFunction->Identity,PlotRange->{-3,6}];


Show[pl1,pl2,pl3,DisplayFunction->$DisplayFunction,PlotRange->{-3,6},AspectRatio->A
utomatic];
```



10, 11, 12 Sehnensatz, Tangentensatz, Sekantensatz

```
Remove["Global`*"]

{xM,yM}={4,3};   r=2;   pM={xM,yM};
k[x_,y_]:= ({x,y}-{xM,yM}).({x,y}-{xM,yM})-r^2;
p0={-4,2};
k[{x_,y_}]:=k[x,y];

polare[x_,y_]:= ({x,y}-{xM,yM}).(p0-{xM,yM})-r^2;
polare[x,y]
```

$-1-8\,(-4+x)-y$

```
solv7=Solve[{polare[x,y]==0,0==k[x,y]},{x,y}]
```

$$\left\{\left\{x\to\frac{2}{65}\left(114-\sqrt{61}\right),\ y\to\frac{1}{65}\left(191+16\sqrt{61}\right)\right\},\ \left\{x\to\frac{2}{65}\left(114+\sqrt{61}\right),\ y\to\frac{1}{65}\left(191-16\sqrt{61}\right)\right\}\right\}$$

```
p1={x,y}/.solv7[[1]]
```

$$\left\{\frac{2}{65}\left(114-\sqrt{61}\right),\ \frac{1}{65}\left(191+16\sqrt{61}\right)\right\}$$

```
N[%]
```

{3.26738, 4.86098}

```
zentrale[t_]:= p0+ t (pM-p0);
zentrale[t]
```

$\{-4+8\,t,\ 2+t\}$

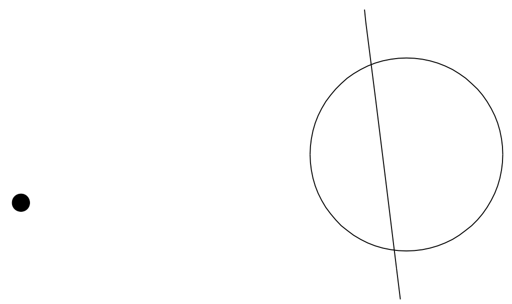
$\{-4+8\,t,\ 2+t\}$

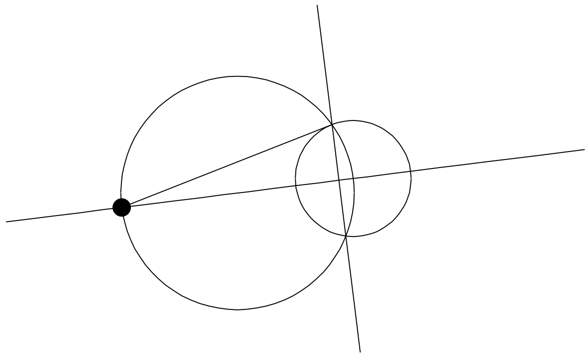
```
pPolM=(pM+p0)/2; rM=Norm[pPolM-p0];
kM[x_,y_]:= ({x,y}-pPolM).({x,y}-pPolM)-rM^2;
```

```
<< Graphics`ImplicitPlot`;
pl1 = ImplicitPlot[k[x,y] == 0, {x, 2,8},DisplayFunction->Identity];
pl2 = ImplicitPlot[polare[x,y] == 0, {x,
2,8},DisplayFunction->Identity,PlotRange->{0,6}];
pl3 = ParametricPlot[zentrale[t], {t,
-0.5,2},DisplayFunction->Identity,PlotRange->{-3,9}];
pl4 = ImplicitPlot[kM[x,y]==0, {x,
-5,8},DisplayFunction->Identity,PlotRange->{-3,9}];
```

```
Show[Graphics[{PointSize[0.03],Point[p0]}],pl1,pl2,DisplayFunction->$DisplayFunctio
n,PlotRange->{0,6},AspectRatio->Automatic];
```

```
Show[Graphics[{PointSize[0.03],Point[p0],Line[{p0,p1}]}],pl1,pl2,pl3,pl4,DisplayFun
ction->$DisplayFunction,PlotRange->{-3,9},AspectRatio->Automatic];
```





Anwendung: Läge der Tangente von p0 nach p1

```
Sqrt[k[p0]]
```

$\sqrt{61}$

```
N[%]
```

7.81025

Anwendung: Läge der Tangente von p0 nach p1

Eine Gerade von p0 aus schneidet den Kreis k in x=5. y ist die grössere der beiden Lösungen. Wo liegt der zweite Schnittpunkt dieser Geraden mit dem Kreis?

```
solv7=Solve[k[5,y]==0,{y}]//Flatten
```

$\{y \to 3 - \sqrt{3}, y \to 3 + \sqrt{3}\}$

```
N[%]
```

$\{y \to 1.26795, y \to 4.73205\}$

```
p2={5,y}/.solv7[[2]]
```

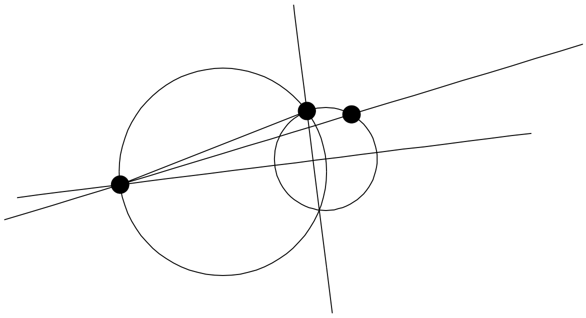
$\{5, 3 + \sqrt{3}\}$

```
g[t_]:=p0+t(p2-p0);
pl5 = ParametricPlot[g[t], {t,
-0.5,2},DisplayFunction->Identity,PlotRange->{-3,9}];
```

```
((p2-p0).(g[t]-p0)//Simplify)== k[p0]
```

$(85 + 2\sqrt{3})\, t == 61$

```
Show[Graphics[{PointSize[0.03],Point[p0],Point[p1],Point[p2],Line[{p0,p1}]}],pl1,pl
2,pl3,pl4,pl5,DisplayFunction->$DisplayFunction,PlotRange->{-3,9},AspectRatio->Auto
matic];
```



```
solve8=Solve[(p2-p0).(p0+t(p2-p0)-p0)== k[p0],{t}]//Flatten
```

$\left\{t \to \frac{61}{85 + 2\sqrt{3}}\right\}$

```
N[%]
```

{t → 0.689545}

```
k[p0]
```

61

```
p3=p0+t(p2-p0)/.solve8
```

$\left\{-4 + \frac{549}{85 + 2\sqrt{3}},\ 2 + \frac{61\left(1 + \sqrt{3}\right)}{85 + 2\sqrt{3}}\right\}$

```
N[%]
```

{2.20591, 3.88387}

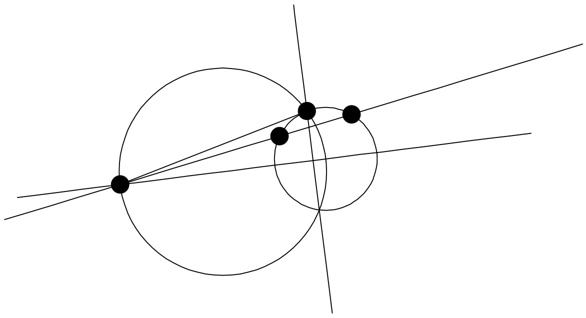
```
Sqrt[(p0-p1).(p0-p1)]//N
```

7.81025

```
{p0,p1,p2,p3}//N
```

{{-4., 2.}, {3.26738, 4.86098}, {5., 4.73205}, {2.20591, 3.88387}}

```
Show[Graphics[{PointSize[0.03],Point[p0],Point[p1],Point[p2],Point[p3],Line[{p0,p1}
]}],pl1,pl2,pl3,pl4,pl5,DisplayFunction->$DisplayFunction,PlotRange->{-3,9},AspectR
atio->Automatic];
```



13, 14 Kegel und Zylinder

a Beispiel Kegel: Schnittgebilde mit der Grundebene?

```
Remove["Global`*"]
```

Sei gegeben Richtungsvektor a, Scheitel sS={3,2,4}; P0

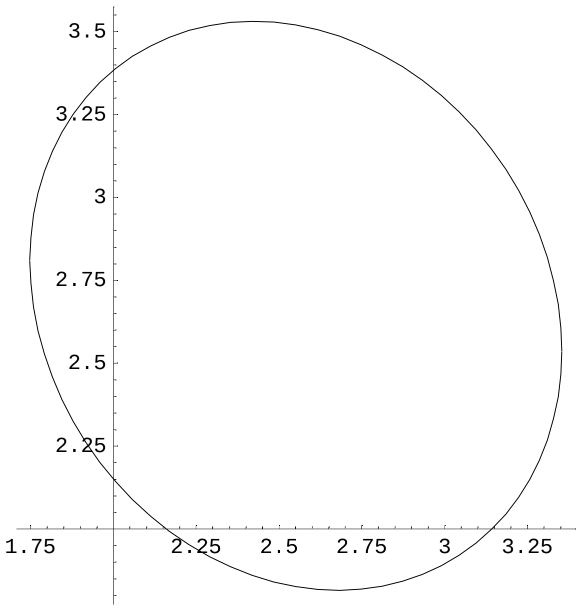
```
a = {4,-6,14};  sS = {3,2,1};  P0 = sS+a+{4,8,0}; P0
```

{11, 4, 15}

```
kegel[x_,y_,z_]:= (a.(sS-P0))^2 (({x,y,z}-sS).({x,y,z}-sS)) - (a.({x,y,z}-sS))^2
((sS-P0).(sS-P0));
kegel[x,y,0] // Simplify
```

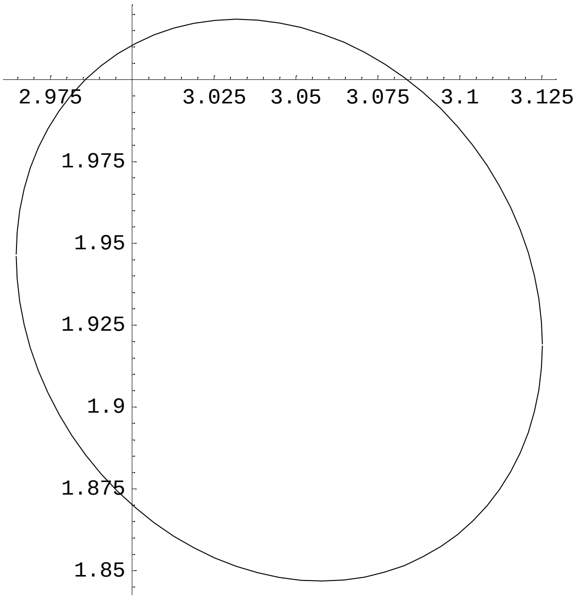
$96\,(6265 + 442\,x^2 - 2406\,y + 387\,y^2 + 4\,x\,(-652 + 33\,y))$

```
<< Graphics`ImplicitPlot`;
ImplicitPlot[Evaluate[kegel[x,y,0] == 0], {x,-10,10}];
```

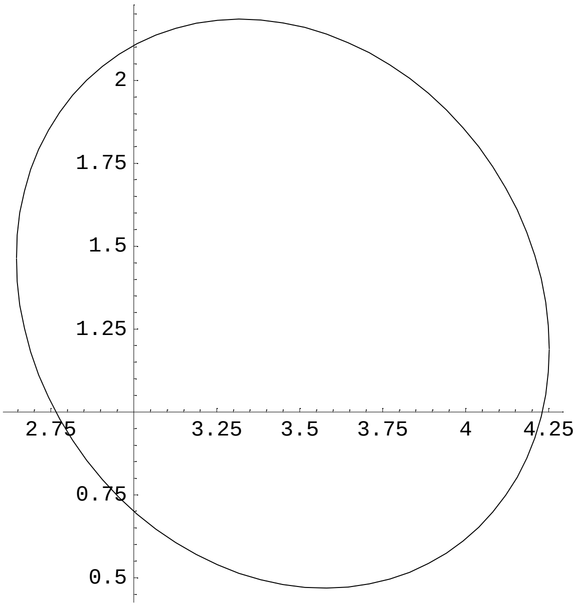


Schnitt verschoben

```
<< Graphics`ImplicitPlot`;
ImplicitPlot[Evaluate[kegel[x,y,1.1] == 0], {x,-10,10}];
```




```
<< Graphics`ImplicitPlot`;
ImplicitPlot[Evaluate[kegel[x,y,2] == 0], {x,-10,10}];
```



a Beispiel Zylinder: Schnittgebilde mit der Grundebene?

```
Remove["Global`*"]
```

Sei gegeben Richtungsvektor a, Scheitel sS={3,2,4}; P0

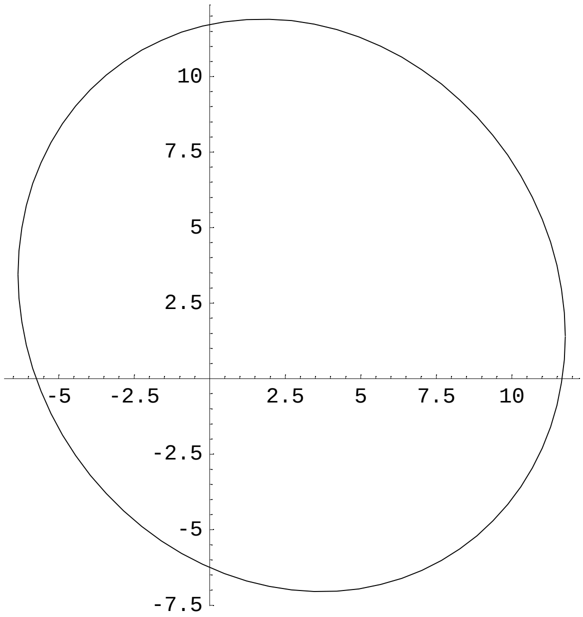
```
a = {4,-6,14};  sS = {3,2,1};  P0 = sS+a+{4,8,0}; P0
```

```
{11, 4, 15}
```

```
zylinder[x_,y_,z_]:= Cross[(sS-P0),a].Cross[(sS-P0),a] -
Cross[({x,y,z}-sS),a].Cross[({x,y,z}-sS),a];
zylinder[x,y,0] // Simplify
```

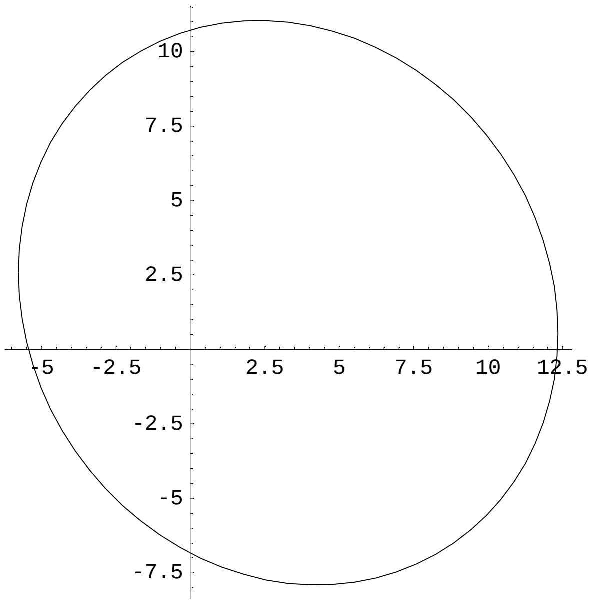
$-4\ (-3885 + 58\ x^2 - 290\ y + 53\ y^2 + 4\ x\ (-86 + 3\ y))$

```
<< Graphics`ImplicitPlot`;
ImplicitPlot[Evaluate[zylinder[x,y,0] == 0], {x,-10,12}];
```



Schnitt verschoben

```
<< Graphics`ImplicitPlot`;
ImplicitPlot[Evaluate[zylinder[x,y,2] == 0], {x,-8,14}];
```



Teil 2: Matrizen und Eigenwertprobleme

1 Matrizenrechnung

```
Remove["Global`*"]
```

a

```
A = {{1,2,3},{3,5,6},{4,5,6}}; A//MatrixForm
```

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 5 & 6 \\ 4 & 5 & 6 \end{pmatrix}$$

```
A1=Inverse[A]; A1 //MatrixForm
```

$$\begin{pmatrix} 0 & -1 & 1 \\ -2 & 2 & -1 \\ \frac{5}{3} & -1 & \frac{1}{3} \end{pmatrix}$$

```
Det[A]
```

-3

```
Det[A1]
```

$-\frac{1}{3}$

b

```
e1={1,0,0}; e2={0,1,0}; e3={0,0,1};
A.e1
```

$\{1, 3, 4\}$

```
A.e2
```

$\{2, 5, 5\}$

```
A.e3
```

$\{3, 6, 6\}$

Spalten von A

c

```
A1.e1
```

$\left\{0, -2, \frac{5}{3}\right\}$

```
A1.e2
```

{-1, 2, -1}

```
A1.e3
```

$\{1, -1, \frac{1}{3}\}$

Spalten der Inversen von A

2 Eigenwerte, Eigenvektoren

```
Remove["Global`*"]
```

a

```
A = {{1, 2}, {2, 3}}; A // MatrixForm
```

$$\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$$

```
Det[A]
```

-1

b1

```
A1=Inverse[A]; A1//MatrixForm
```

$$\begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix}$$

```
EM=IdentityMatrix[2]; EM//MatrixForm
```

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

```
Det[A-λ EM]
```

$-1 - 4 \lambda + \lambda^2$

A xVec = xVec= EM xVec bedeutet (A- EM) xVec = 0Vec und dass hier nicht nur die Null-Lösung existiert. Das heisst Det[A- EM]=0. Diese Gleichung ist lösbar (charakteristische Gleichung)!

```
solv1=Solve[Det[A-λ EM]==0,{λ}]//Flatten
```

$\{\lambda \to 2 - \sqrt{5}, \lambda \to 2 + \sqrt{5}\}$

```
N[%]
```

$\{\lambda \to -0.236068, \lambda \to 4.23607\}$

```
λ1 = λ/.solv1[[1]]
```

$2-\sqrt{5}$

```
λ2 = λ/.solv1[[2]]
```

$2+\sqrt{5}$

b2

```
solv2=Solve[A.{x,y}==λ1{x,y},{x,y}]//Flatten
```

$\{x \to -\frac{1}{2}(1+\sqrt{5})\,y\}$

```
x1Vec = ({x,y}/.solv2)/.y->1
```

$\{\frac{1}{2}(-1-\sqrt{5}),\,1\}$

```
Simplify[A.x1Vec] == Simplify[λ1 x1Vec]
```

True

b3

```
N[λ1]
```

-0.236068

```
solv2a=Solve[A.{x,y}== -0.236068 {x,y},{x,y}]//Flatten
```

$\{x \to 0.,\, y \to 0.\}$

b4

```
solv3=Solve[A.{x,y}==λ2{x,y},{x,y}]//Flatten
```

$\{x \to -\frac{1}{2}(1-\sqrt{5})\,y\}$

```
x2Vec = ({x,y}/.solv3)/.y->1
```

$\{\frac{1}{2}(-1+\sqrt{5}),\,1\}$

```
Simplify[A.x2Vec] == Simplify[λ2 x2Vec]
```

True

b5

```
N[λ2]
```

4.23607

```
Solve[A.{x,y}==4.23607 {x,y},{x,y}]//Flatten
```

$\{x \to 0., y \to 0.\}$