Lösungen

Beispiele von Aufgaben mit Laplace-Transformation

Hier werden nur Lösungswege für die folgenden Aufgaben aufgezeigt:
4.2 // II/10 Aufgaben 1 und 2 sowie 3.8 // II/18, Aufgabe 2

1:

1.1. Linke Seite transformieren, Anfangswerte anpassen

```
links = LaplaceTransform[4 y''[t]-y[t],t,s] /. {LaplaceTransform[y[t],t,s]→Y[s],y[0]→1,y
```

$-Y[s] + 4\left(1 - s + s^2\, Y[s]\right)$

1.2. Rechte Seite transformieren

```
rechts=LaplaceTransform[t ,t,s]
```

$\frac{1}{s^2}$

1.3. Gleichung links = rechts lösen

```
solv=Solve[links==rechts,{Y[s]}] // Flatten
```

$\left\{Y[s] \to \frac{1 - 4 s^2 + 4 s^3}{s^2 \left(-1 + 4 s^2\right)}\right\}$

1.4. Rücktransformation

```
U[s]:=Y[s]/. solv;  U[s]
```

$\frac{1 - 4 s^2 + 4 s^3}{s^2 \left(-1 + 4 s^2\right)}$

```
InverseLaplaceTransform [U[s], s, t]
```

$\frac{e^{-t/2}}{2} + \frac{e^{t/2}}{2} - t$

```
U[s]//Apart
```

$-\frac{1}{s^2} + \frac{1}{-1 + 2 s} + \frac{1}{1 + 2 s}$

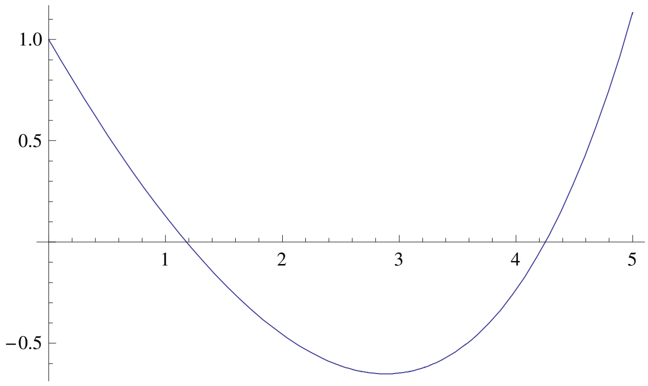
```
u0[t_]:=InverseLaplaceTransform[U[s],s,t]//Simplify; u0[t]
```

$$\frac{1}{2}\left(e^{-t/2}+e^{t/2}-2\,t\right)$$

```
% // N
```

$$0.5\left(2.71828^{-0.5\,t}+2.71828^{0.5\,t}-2.\,t\right)$$

```
Plot[Evaluate[{u0[t]}],{t,0,5}]
```



2:

■ Alles wieder löschen, sauber machen

```
Remove["Global`*"]
```

Remove::rmnsm : There are no symbols matching "Global`*".

■ 2.1. Linke Seite transformieren, Anfangswerte anpassen

```
links = LaplaceTransform[y'''[t]-y[t],t,s] /. {LaplaceTransform[y[t],t,s]→Y[s],y[0]→0,y'
```

$-1 - Y[s] + s^3\,Y[s]$

■ 2.2. Rechte Seite transformieren

```
rechts=LaplaceTransform[DiracDelta[t] ,t,s]
```

1

■ 2.3. Gleichung links = rechts lösen

```
solv=Solve[links==rechts,{Y[s]}] // Flatten
```

$$\left\{Y[s]\to\frac{2}{-1+s^3}\right\}$$

■ 2.4. Rücktransformation

```
U[s]:=Y[s]/. solv;  U[s]
```

$$\frac{2}{-1+s^3}$$

```
InverseLaplaceTransform [U[s], s, t] // TrigFactor // Expand
```

$$\frac{2\,e^t}{3} - \frac{4}{3}\,e^{-t/2}\,\mathrm{Sin}\left[\frac{\pi}{6} + \frac{\sqrt{3}\,t}{2}\right]$$

```
U[s]//Apart
```

$$\frac{2}{3\,(-1+s)} - \frac{2\,(2+s)}{3\,\left(1+s+s^2\right)}$$

```
u0[t_]:=InverseLaplaceTransform[U[s],s,t]//Simplify; u0[t]
```

$$-\frac{2}{3}\,e^{-t/2}\left(-e^{3t/2} + \mathrm{Cos}\left[\frac{\sqrt{3}\,t}{2}\right] + \sqrt{3}\,\mathrm{Sin}\left[\frac{\sqrt{3}\,t}{2}\right]\right)$$

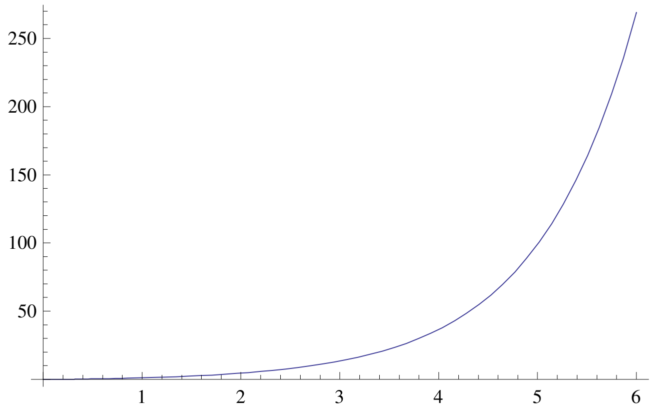
```
% // N
```

$$-0.666667\;2.71828^{-0.5\,t}\left(-1.\;2.71828^{1.5\,t} + \mathrm{Cos}[0.866025\,t] + 1.73205\,\mathrm{Sin}[0.866025\,t]\right)$$

```
% // Expand
```

$$0.666667\;2.71828^{1.\,t} - 0.666667\;2.71828^{-0.5\,t}\,\mathrm{Cos}[0.866025\,t] - 1.1547\;2.71828^{-0.5\,t}\,\mathrm{Sin}[0.866025\,t]$$

```
Plot[Evaluate[{u0[t]}],{t,0,6}]
```



3:

■ Alles wieder löschen, sauber machen[t]

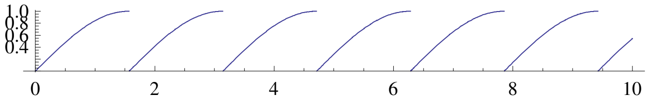
```
Remove["Global`*"]

f[t_] := Sin[t] /; 0 ≤ t && t < Pi / 2;
f[t_] := Sin[t - Pi / 2 Floor[2 t / Pi]]; ?f
```

```
Global`f
```

$$f[t\_] := \text{Sin}[t] \;/;\; 0 \le t \;\&\&\; t < \frac{\pi}{2}$$

$$f[t\_] := \text{Sin}\left[t - \frac{1}{2}\pi\,\text{Floor}\left[\frac{2t}{\pi}\right]\right]$$

```
Plot[f[t], {t, 0, 10}, AspectRatio → Automatic]
```




```
LaplaceTransform[f[t], t, s]
```

$$\text{LaplaceTransform}\left[\text{Sin}\left[t - \frac{1}{2}\pi\,\text{Floor}\left[\frac{2t}{\pi}\right]\right], t, s\right]$$

```
rechts = 1 / (1 - E^(-s Pi / 2)) Integrate[E^(-s t) Sin[t], {t, 0, Pi / 2}] // Simplify
```

$$\frac{e^{\frac{\pi s}{2}} - s}{\left(-1 + e^{\frac{\pi s}{2}}\right)\left(1 + s^2\right)}$$

■ 3.1. Linke Seite transformieren, Anfangswerte anpassen

```
links = LaplaceTransform[y'[t]-y[t],t,s] /. {LaplaceTransform[y[t],t,
s]→Y[s],y[0]→1}
```

$$-1 - Y[s] + s\,Y[s]$$

■ 3.2. Rechte Seite transformieren

■ 3.3. Gleichung links = rechts lösen

```
solv=Solve[links==rechts,{Y[s]}] // Flatten
```

$$\left\{Y[s] \to \frac{-1 + 2\,e^{\frac{\pi s}{2}} - s - s^2 + e^{\frac{\pi s}{2}}\,s^2}{\left(-1 + e^{\frac{\pi s}{2}}\right)(-1 + s)\left(1 + s^2\right)}\right\}$$

■ 3.4. Rücktransformation

```
U[s]:=Y[s]/. solv;  U[s]
```

$$\frac{-1 + 2\,e^{\frac{\pi s}{2}} - s - s^2 + e^{\frac{\pi s}{2}}\,s^2}{\left(-1 + e^{\frac{\pi s}{2}}\right)(-1 + s)\left(1 + s^2\right)}$$

```
U[s]//Apart
```

$$\frac{1}{\left(-1 + e^{\frac{\pi s}{2}}\right)\left(-1 - s^2\right)} + \frac{2 + s^2}{-1 + s - s^2 + s^3}$$

```
Apart[(2 + s^2) / (-1 + s - s^2 + s^3)]
```

$$\frac{3}{2\,(-1 + s)} + \frac{-1 - s}{2\left(1 + s^2\right)}$$

```
Limit[s U[s], s → 0]
```

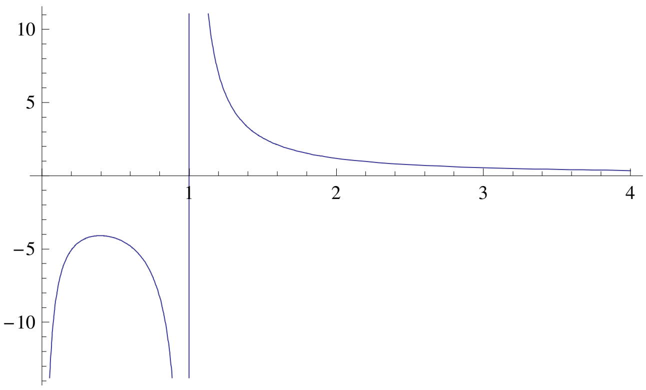
$-\frac{2}{\pi}$

```
u0[t_]:=InverseLaplaceTransform[U[s],s,t]//Simplify; u0[t]
```

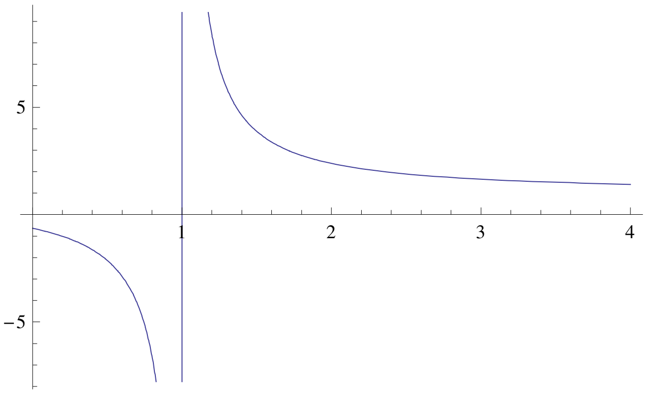
$$-\text{InverseLaplaceTransform}\left[\frac{1+s+s^2-e^{\frac{\pi s}{2}}\left(2+s^2\right)}{\left(-1+e^{\frac{\pi s}{2}}\right)\left(-1+s-s^2+s^3\right)}, s, t\right]$$

```
(* Plot[Evaluate[{u0[t]}],{t,0,4}] geht nicht *)
```

```
Plot[Evaluate[U[s]], {s, 0, 4}]
```




```
Plot[Evaluate[s U[s]], {s, 0, 4}]
```



4:

■ Alles wieder löschen, sauber machen

```
Remove["Global`*"]
```

■ 4.1. Linke Seiten transformieren, Anfangswerte anpassen

```
links1 = LaplaceTransform[y'[t]+z'[t],t,s] /. {LaplaceTransform[y[t],t,s]→Y[s],
LaplaceTransform[z[t],t,s]→Z[s],y[0]→0,z[0]→0}
```

$s\,Y[s] + s\,Z[s]$

```
links2 = LaplaceTransform[y[t]+2 z[t]-2y'[t]-z'[t],t,s] /.
{LaplaceTransform[y[t],t,s]→Y[s],LaplaceTransform[z[t],t,s]→Z[s],y[0]→0,z[0]→0}
```

Y[s] - 2 s Y[s] + 2 Z[s] - s Z[s]

■ 4.2. Rechte Seite transformieren

```
us[t_]:=UnitStep[t]-UnitStep[t-1];
Plot[us[t],{t,-1,3},AspectRatio->Automatic];
```

```
rechts1=LaplaceTransform[0 ,t,s]
```

0

```
rechts2=LaplaceTransform[us[t] ,t,s]
```

$$\frac{1}{s} - \frac{e^{-s}}{s}$$

■ 4.3. Gleichungssystem links1 = rechts1, links2 = rechts2 lösen

```
solv=Solve[{links1==rechts1,links2==rechts2},{Y[s],Z[s]}] // Flatten
```

$$\left\{Y[s] \to -\frac{e^{-s}\,(-1+e^{s})}{s\,(1+s)},\ Z[s] \to \frac{e^{-s}\,(-1+e^{s})}{s\,(1+s)}\right\}$$

■ 4.4. Rücktransformation

```
Uy[s]:=Y[s]/. solv;  Uy[s]
```

$$-\frac{e^{-s}\,(-1+e^{s})}{s\,(1+s)}$$

```
Uy[s]//Apart
```

$$-\frac{1}{s\,(1+s)} + \frac{e^{-s}}{s\,(1+s)}$$

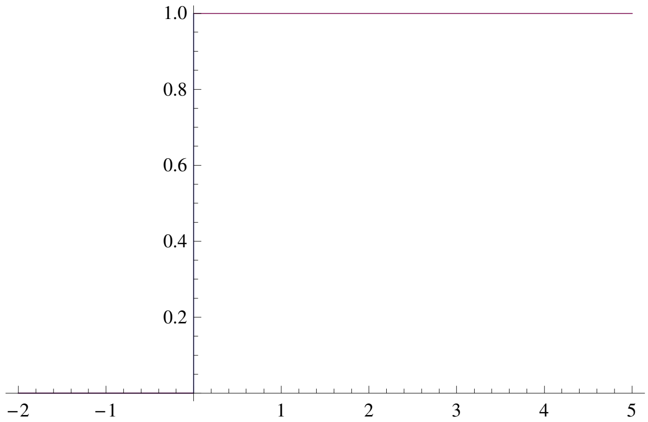
```
uy0[t_]:=InverseLaplaceTransform[Uy[s],s,t]//Simplify; uy0[t]
```

$$-e^{-t}\left(-1+e^{t}+\left(e-e^{t}\right)\,\text{HeavisideTheta}[-1+t]\right)$$

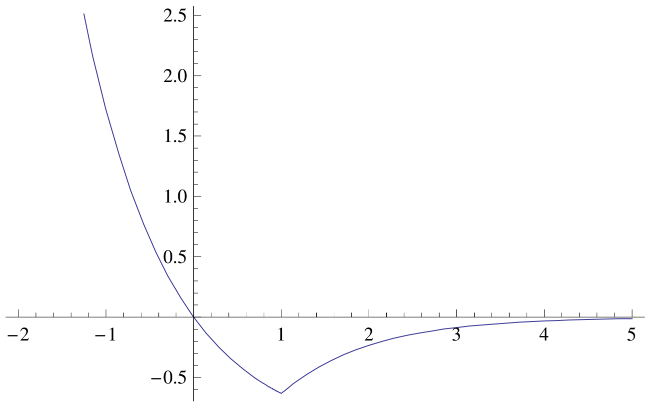
```
uy0[t] // Expand
```

$$-1+e^{-t}+\text{HeavisideTheta}[-1+t]-e^{1-t}\,\text{HeavisideTheta}[-1+t]$$

```
Plot[{HeavisideTheta[t], UnitStep[t]}, {t, -2, 5}]
```



```
Plot[uy0[t], {t, -2, 5}]
```



```
Uz[s]:=Z[s]/. solv;  Uz[s]
```

$$\frac{e^{-s}(-1+e^{s})}{s(1+s)}$$

```
Uz[s]//Apart
```

$$\frac{1}{s(1+s)}-\frac{e^{-s}}{s(1+s)}$$

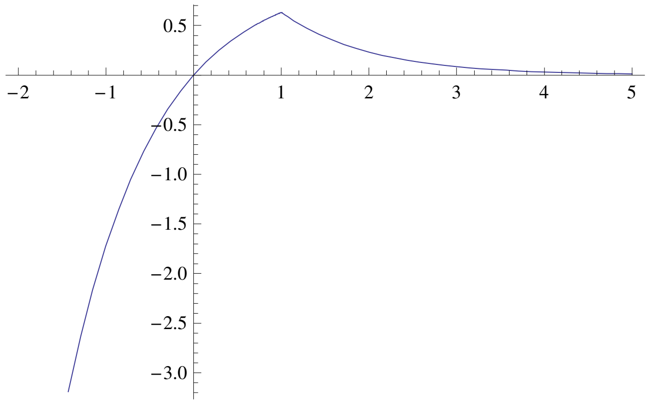
```
uz0[t_]:=InverseLaplaceTransform[Uz[s],s,t]//Simplify; uz0[t]
```

$e^{-t}\left(-1+e^{t}+\left(e-e^{t}\right)\text{HeavisideTheta}[-1+t]\right)$

```
uz0[t] // Expand
```

$1-e^{-t}-\text{HeavisideTheta}[-1+t]+e^{1-t}\,\text{HeavisideTheta}[-1+t]$

```
Plot[uz0[t], {t, -2, 5}]
```



5: Berechnung von Krümmung und Anschmiegekreis bei ebenen Kurven

5.1. Definition der Funktion

```
Remove["Global`*"]
```

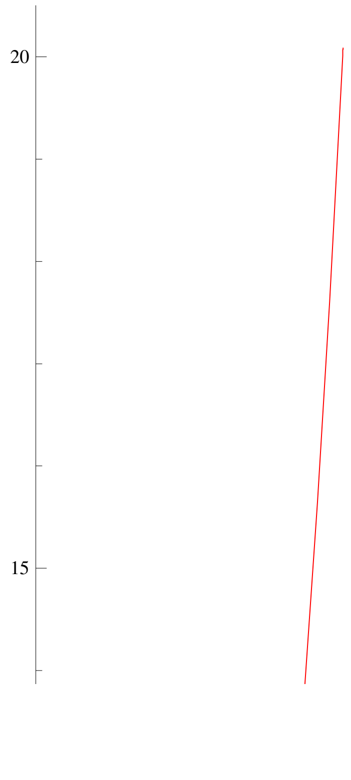
```
fx[t_] := t;  fy[t_] := E^t;  f[t_] := {fx[t], fy[t]};
```

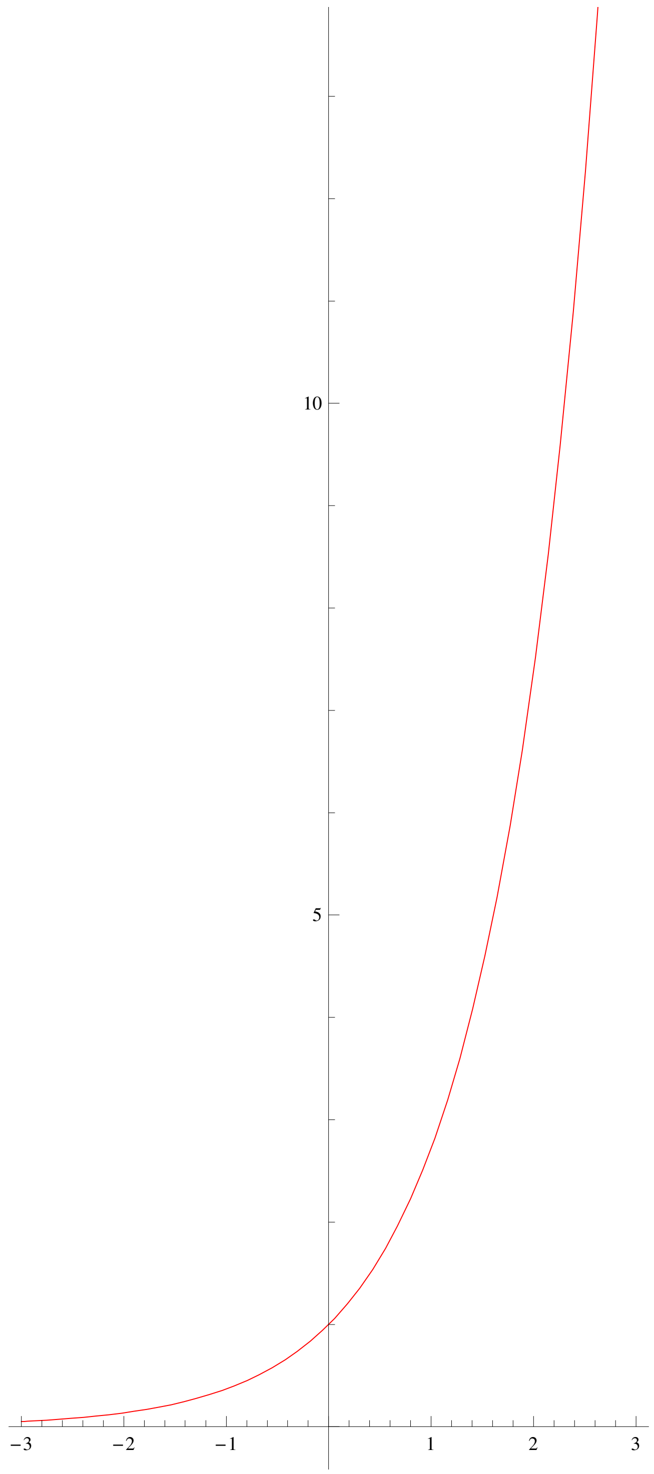
5.2. Definition des Bereiches der Variablen

```
tmin = -2;  tmax = 2;
```

5.3. Plot Ausgangsfunktion

```
origPlot =
 ParametricPlot[Evaluate[f[t]], {t, tmin - 1, tmax + 1}, PlotStyle → {RGBColor[1, 0, 0]}]
```






```
tmin = -1;  tmax = 1.5;
```

■ 5.4. Krümmung

```
vLen[v_] := Sqrt[v.v]
```

```
f3d[t_] := Join[f[t], {0}]; f3d[t]
```

$\{t, e^t, 0\}$

```
Cross[D[f3d[t], t], D[f3d[t], {t, 2}]]
```

$\{0, 0, e^{t}\}$

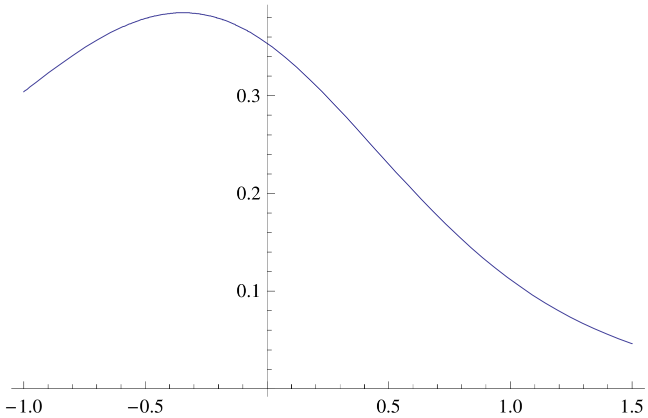
```
Cross[f3d'[t], f3d''[t]]
```

$\{0, 0, e^{t}\}$

```
x[t_] = (Cross[D[f3d[t], t], D[f3d[t], {t, 2}]] / vLen[D[f3d[t], t]]^3)[[3]];  x[t]
```

$$\frac{e^{t}}{\left(1 + e^{2t}\right)^{3/2}}$$

```
Plot[Evaluate[x[t]], {t, tmin, tmax}]
```

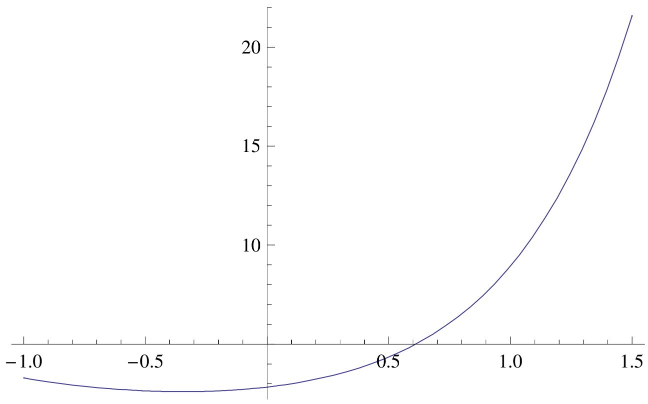


■ 5.5. Krümmungsradius

```
ρ[t_] := Abs[1 / x[t]];  ρ[t]
```

$e^{-\text{Re}[t]} \text{Abs}\left[1 + e^{2t}\right]^{3/2}$

```
Plot[Evaluate[ρ[t]], {t, tmin, tmax}]
```



■ 5.6. Krümmungskreismittelpunkt (Evolute)

```
Take[Cross[Cross[f3d'[t], f3d''[t]], f3d'[t]] /
  vLen[Cross[Cross[f3d'[t], f3d''[t]], f3d'[t]]]]
```

$$\left\{-\frac{e^{2t}}{\sqrt{e^{2t} + e^{4t}}}, \frac{e^{t}}{\sqrt{e^{2t} + e^{4t}}}, 0\right\}$$

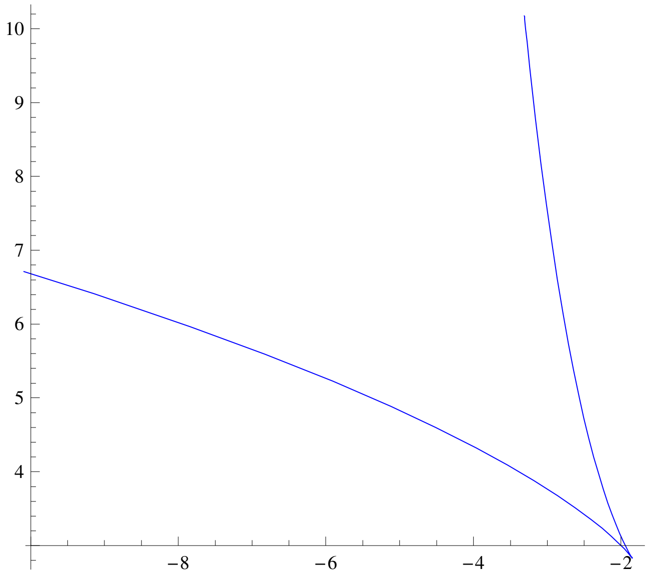
```
n[t_] = Take[Cross[Cross[f3d'[t], f3d''[t]], f3d'[t]] /
   vLen[Cross[Cross[f3d'[t], f3d''[t]], f3d'[t]]], 2]; n[t]
```

$$\left\{-\frac{e^{2t}}{\sqrt{e^{2t}+e^{4t}}},\ \frac{e^{t}}{\sqrt{e^{2t}+e^{4t}}}\right\}$$

```
m[t_] := f[t] + ρ[t] * n[t]; m[3]
```

$$\left\{3-\frac{e^{3}\left(1+e^{6}\right)^{3/2}}{\sqrt{e^{6}+e^{12}}},\ e^{3}+\frac{\left(1+e^{6}\right)^{3/2}}{\sqrt{e^{6}+e^{12}}}\right\}$$

```
evolute = ParametricPlot[Evaluate[m[t]], {t, tmin - 1.3, tmax},
  AspectRatio -> Automatic, PlotStyle → RGBColor[0, 0, 1]]
```



■ 5.7. Ursprüngliche Funktion mit den Schmiegekreisen

```
myCirc[t_] := Circle[m[t], ρ[t]]; myCirc[t]
```

$$\text{Circle}\left[\left\{t-\frac{e^{2t-\text{Re}[t]}\,\text{Abs}\left[1+e^{2t}\right]^{3/2}}{\sqrt{e^{2t}+e^{4t}}},\ e^{t}+\frac{e^{t-\text{Re}[t]}\,\text{Abs}\left[1+e^{2t}\right]^{3/2}}{\sqrt{e^{2t}+e^{4t}}}\right\},\ e^{-\text{Re}[t]}\,\text{Abs}\left[1+e^{2t}\right]^{3/2}\right]$$

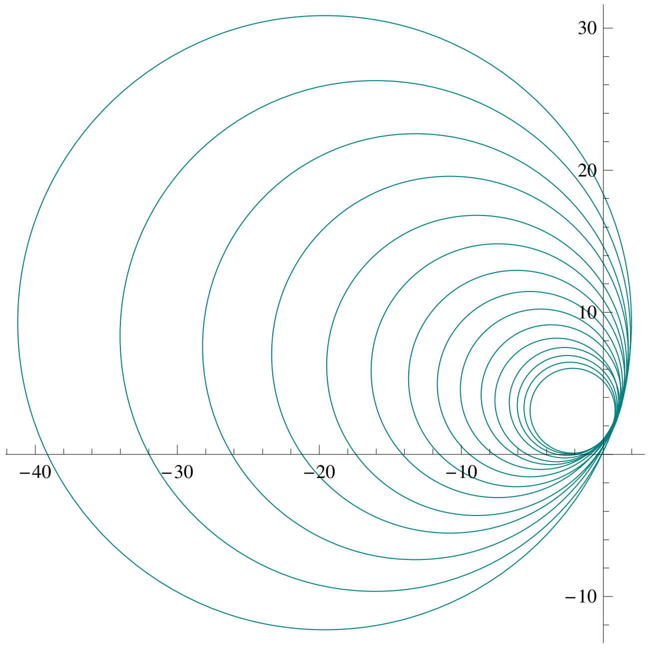
```
myCirc[2]
```

$$\text{Circle}\left[\left\{2-\frac{e^{2}\left(1+e^{4}\right)^{3/2}}{\sqrt{e^{4}+e^{8}}},\ e^{2}+\frac{\left(1+e^{4}\right)^{3/2}}{\sqrt{e^{4}+e^{8}}}\right\},\ \frac{\left(1+e^{4}\right)^{3/2}}{e^{2}}\right]$$

```
myPlotTable = Table[Evaluate[myCirc[t]], {t, 0.1, tmax, 0.1}]
```

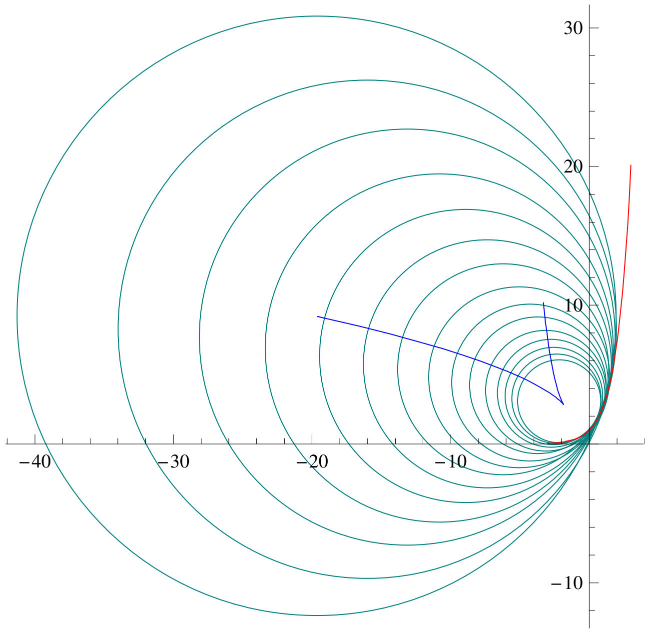
```
{Circle[{-2.1214, 3.11518}, 2.99579],
 Circle[{-2.29182, 3.26154}, 3.22046], Circle[{-2.52212, 3.44054}, 3.51216],
 Circle[{-2.82554, 3.65397}, 3.88317], Circle[{-3.21828, 3.90397}, 4.34877],
 Circle[{-3.72012, 4.19305}, 4.92795], Circle[{-4.3552, 4.52409}, 5.64419],
 Circle[{-5.15303, 4.90041}, 6.52637], Circle[{-6.14965, 5.32578}, 7.61002],
 Circle[{-7.38906, 5.80444}, 8.93872], Circle[{-8.92501, 6.3412}, 10.5658],
 Circle[{-10.8232, 6.94143}, 12.5567], Circle[{-13.1637, 7.61113}, 14.9913],
 Circle[{-16.0446, 8.357}, 17.9672], Circle[{-19.5855, 9.18651}, 21.6041]}
```

```
constrPlot = Show[Graphics[Join[{RGBColor[0, 0.5, 0.5]}, myPlotTable]],
  Axes → True, AspectRatio → Automatic]
```



5.8. Alles zusammen

```
Show[constrPlot, evolute, origPlot]
```



5.9. Krümmungskreismittelpunkt für t = 0:

```
m[0]
```

```
{-2, 3}
```